Braided $\otimes$-product of von Neumann algebras

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Quantum group seminar

(online)

Motivation

A Locally Compact Quantum Group (LCQG)

is a C^*/von Neumann bialgebra (A, Δ), $\Delta \in \mathrm{Mor}(A, A \otimes A)$

① defined through a modular multiplicative unitary

(Woronowicz, Sołtan) $W \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H})$, $A \subseteq \mathcal{B}(\mathcal{H})$

$\Uparrow$

② admitting invariant weights

(Kustermans-Vaes)

Motivation

A braided Locally Compact Quantum Group (over qtriang. $(\mathbb{G}, R)$)
is a braided C^*-bialgebra (A, Δ), $\mathbb{G} \curvearrowright A$, $\Delta \in \mathrm{Mor}(A, A \boxtimes A)$

① defined through a braided modular multiplicative unitary

(Meyer, Roy, Woronowicz)

$W \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H})$, $A \subseteq \mathcal{B}(\mathcal{H})$

$X \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H})$ $\quad \mathbb{G} \curvearrowright \mathcal{H}$

$\Uparrow$

② ?? $\boxed{\text{Braided } \boxtimes \text{ for von Neumann algebras?}}$

Invariant weights ?

<u>Note</u>: well-developed algebraic theory (Majid, Heckenberger, Schneider, ...)

1. Quasitriangular von Neumann algebraic quantum groups

Def: A quasi-triangular LCQG $(\mathbb{G}, \hat{R})$ is

① LCQG $\mathbb{G}$, so $(L^\infty(\mathbb{G}), \Delta)$, $(L(\mathbb{G}), \hat{\Delta})$

② unitary bicharacter $\hat{R} \in L(\mathbb{G}) \bar{\otimes} L(\mathbb{G})$,

so $$(\hat{\Delta} \otimes id)\hat{R} = \hat{R}_{13} \hat{R}_{23}$$

$$(id \otimes \hat{\Delta})\hat{R} = \hat{R}_{13} \hat{R}_{12}$$

③ $\hat{R}$ twists between $\hat{\Delta}$ and $\hat{\Delta}^{op}$:

$$\hat{R} \hat{\Delta}(\ ) \hat{R}^* = \hat{\Delta}^{op}(\ )$$

E.g.: ① $\mathbb{H}$ LCQG $\Rightarrow$ Drinfeld double $\mathbb{G} = D\mathbb{H}$ is naturally q-triang.

② $\mathbb{G} = \mathbb{R}$, $\hat{R} = e^{2\pi i \hbar (X \otimes X)} \in L(\mathbb{R}) \bar{\otimes} L(\mathbb{R})$, $\hbar \in \mathbb{R}$.

<u>Thm</u>: $(\mathbb{G}, \hat{R})$ q. triang. q. group

$\Longrightarrow \mathrm{Rep}_u(\mathbb{G})$ unitarily braided.

<u>Pf</u>: $\hat{R} \in L(\mathbb{G}) \bar{\otimes} L(\mathbb{G}) \underset{\text{(Kustermans)}}{\Longrightarrow} \hat{R}_u \in C^*(\mathbb{G}) \otimes C^*(\mathbb{G})$

Then $(\mathrm{id}, \times) : (\mathrm{Rep}(\mathbb{G}), \otimes) \cong (\mathrm{Rep}(\mathbb{G}), \otimes^{op})$

$$\overset{\mathcal{H}\quad\ \mathcal{G}}{\times} = (\pi_{\mathcal{H}} \otimes \pi_{\mathcal{G}})(\hat{R}_u)\underbrace{\Sigma}_{\text{ordinary flip}} : \mathcal{H} \otimes \mathcal{G} \longrightarrow \mathcal{G} \otimes \mathcal{H}$$

$\blacksquare$

2. Braided $\otimes$-product of vN algebras

M vN algebra : $\mathbb{G} \curvearrowright M$ via $\alpha_M : M \to L^\infty(\mathbb{G}) \bar{\otimes} M$

$\Rightarrow \exists$ Covariant model $\mathbb{G} \curvearrowright (\mathcal{H}, M)$: $M \subseteq B(\mathcal{H})$, $\alpha(x) = U_{\mathcal{H}}^* (1 \otimes x) U_{\mathcal{H}}$

<u>Thm</u> : $(\mathbb{G}, \hat{R})$ q. triang, $\mathbb{G} \curvearrowright (\mathcal{H}, M)$, $\mathbb{G} \curvearrowright (\mathcal{G}, N)$

$\Rightarrow$ ① vN algebra $M \bar{\boxtimes} N := \overline{\operatorname{span}}^{w} \{ X (1 \otimes M) X^* (1 \otimes N) \}$

② $\mathbb{G} \curvearrowright M \bar{\boxtimes} N$ via $U_{\mathcal{H} \otimes \mathcal{G}}$

③ $M \bar{\boxtimes} N$ and $\alpha_{M \bar{\boxtimes} N}$ independent of $\mathcal{H}, \mathcal{G}$.

<u>Remarks</u> : ① In general $\bar{\boxtimes}$ associative, but $M \bar{\boxtimes} N \not\cong N \bar{\boxtimes} M$.

② General bicharacter : ✓ $M \bar{\boxtimes} N$ ✗ $\alpha_{M \bar{\boxtimes} N}$

<u>Example</u> : $(\mathbb{R}, e^{2\pi i \hbar (X \otimes X)}) \underset{\hbar \neq 0}{\Rightarrow} L^\infty(\mathbb{R}) \bar{\boxtimes} L^\infty(\mathbb{R}) \cong B(L^2(\mathbb{R}))$

vN alg $M \underset{\text{Haagerup '75}}{\Rightarrow} (L^2(M), \pi_M, J, L^2(M)_+)$ (standard construction)

LCQG $\mathbb{G}$ $\rightsquigarrow$ $\mathbb{G} \overset{\alpha}{\curvearrowright} M \underset{\text{Vaes '01}}{\Rightarrow} (L^2(M), U_\alpha)$ (standard implementation)

<u>Thm</u> : $(\mathbb{G}, \hat{R})$ q. triang., $\mathbb{G} \curvearrowright M$, $\mathbb{G} \curvearrowright N$

$$\Rightarrow \begin{cases} L^2(M \bar{\boxtimes} N) \cong L^2(M) \otimes L^2(N) \\ \pi_{M \bar{\boxtimes} N} \cong \chi(1 \otimes \pi_M(-))\chi^*(1 \otimes \pi_N(-)) = \pi_M \boxtimes \pi_N \\ J_{M \bar{\boxtimes} N} \cong (\chi \circ \Sigma) \circ J_M \otimes J_N \\ U_{\alpha_{M \bar{\boxtimes} N}} \cong U_{\alpha_M} \text{\textcircled{T}} U_{\alpha_N} \end{cases}$$

3. Cocycle deformations

Def: Quantum linking groupoid $(L(\mathcal{G}), \hat{\Delta})$:

Linking vN alg. $L(\mathcal{G}) = \begin{pmatrix} L(\mathbb{H}) & L(\mathbb{X}) \\ L(\mathbb{Y}) & L(\mathbb{G}) \end{pmatrix}$

w/ $\hat{\Delta}_{ij} : L(\mathcal{G})_{ij} \longrightarrow L(\mathcal{G})_{ij} \bar{\otimes} L(\mathcal{G})_{ij}$

s.t. $(L(\mathbb{H}), \hat{\Delta}_{11})$, $(L(\mathbb{G}), \hat{\Delta}_{22})$ are LCQG.

E.g.: $\hat{\Omega} \in L(\hat{\mathbb{G}}) \bar{\otimes} L(\hat{\mathbb{G}})$ unitary 2-cocycle:

$$(\hat{\Omega} \otimes 1)(\hat{\Delta} \otimes id)(\hat{\Omega}) = (1 \otimes \hat{\Omega})(id \otimes \hat{\Delta})(\hat{\Omega})$$

$$\Rightarrow L(\mathcal{G}_{\hat{\Omega}}) = \begin{pmatrix} L(\mathbb{G}) & L(\mathbb{G}) \\ L(\mathbb{G}) & L(\mathbb{G}) \end{pmatrix}, \quad \hat{\Delta} = \begin{pmatrix} \hat{\Omega} \hat{\Delta}(-) \hat{\Omega}^* & \hat{\Omega} \hat{\Delta}(-) \\ \hat{\Delta}(-) \hat{\Omega}^* & \hat{\Delta} \end{pmatrix}$$

E.g.: $\hat{\chi} \in L(\mathbb{G}_1) \bar{\otimes} L(\mathbb{G}_2)$ unitary bicharacter

$$\Rightarrow \hat{\Omega} = \hat{\chi}_{32} \in L(\mathbb{G}) \bar{\otimes} L(\mathbb{G}), \qquad \mathbb{G} = \mathbb{G}_1 \times \mathbb{G}_2$$

Thm: $\mathcal{G} = \begin{pmatrix} \mathbb{H} & \mathbb{X} \\ \mathbb{Y} & \mathbb{G} \end{pmatrix}$ quantum linking groupoid

$$\Rightarrow \quad \mathbb{G}\text{-vN-alg} \cong \mathbb{H}\text{-vN-alg}.$$

$$(M, \alpha) \longmapsto (\mathrm{Ind}_{\mathbb{X}}(M), \mathrm{Ind}_{\mathbb{X}}(\alpha))$$

Thm: $(\mathbb{G}, \hat{R})$ q. triang. LCQG $\rightsquigarrow$ associated

$$\mathcal{G}_{\hat{R}_{21}} = \begin{pmatrix} \mathbb{G} \bowtie \mathbb{G} & \mathbb{G} \ltimes \mathbb{G} \\ \mathbb{G} \rtimes \mathbb{G} & \mathbb{G} \times \mathbb{G} \end{pmatrix}$$

Then ① $\hat{\Delta} : L(\mathbb{G}) \to L(\mathbb{G} \bowtie \mathbb{G})$ is LCQG - hom. $\Rightarrow \mathbb{G} \hookrightarrow \mathbb{G} \bowtie \mathbb{G}$

② If $\mathbb{G} \curvearrowright M$, $\mathbb{G} \curvearrowright N$:

$$(M \boxtimes N, \alpha_{M \boxtimes N}) \cong \mathrm{Res}_{\mathbb{G}}(\mathrm{Ind}_{\mathbb{G} \ltimes \mathbb{G}}(M \bar{\otimes} N), \mathrm{Ind}_{\mathbb{G} \ltimes \mathbb{G}}(\alpha_M \otimes \alpha_N))$$

4. C^*-algebraic subtleties

$$\mathcal{G} = \begin{pmatrix} \mathbb{H} & \mathbb{X} \\ \mathbb{Y} & \mathbb{G} \end{pmatrix} \implies C^*(\mathcal{G}) = \begin{pmatrix} C^*(\mathbb{H}) & C^*(\mathbb{X}) \\ C^*(\mathbb{Y}) & C^*(\mathbb{G}) \end{pmatrix}$$

Thm: ① $\exists$ monoidal unitary equivalence

$$(\mathrm{Ind}_{\mathbb{X}}, u) : \mathrm{Rep}_u(\mathbb{G}) \cong \mathrm{Rep}_u(\mathbb{H}), \quad \mathcal{H} \mapsto \mathrm{Ind}_{\mathbb{X}}(\mathcal{H})$$

$$\mathrm{Rep}_u(\mathbb{G}) \cong \mathrm{Rep}_u(\mathcal{G}) \cong \mathrm{Rep}_u(\mathbb{H})$$

$$\mathrm{Ind}_{\mathbb{X}}(\mathcal{H}) = C^*(\mathbb{X}) \underset{C^*(\mathbb{G})}{\otimes} \mathcal{H}$$

② IF $\mathbb{G} \overset{\alpha}{\curvearrowright} M$: $\exists$ canonical unitary intertwiner

$$\left(L^2(\mathrm{Ind}_{\mathbb{X}}(M)), U_{\mathrm{Ind}_{\mathbb{X}}(\alpha)}\right) \cong \left(\mathrm{Ind}_{\mathbb{X}}(L^2(M)), U_{\mathrm{Ind}_{\mathbb{X}}(\alpha)}\right)$$

When is $\mathrm{Ind}_{\mathbb{X}}$ itself trivial?

Def: $\mathbb{G}$ W*-cleft if $\mathcal{G} = \mathcal{G}_{\hat{\Omega}}$, $\hat{\Omega} \in L(\mathbb{G}) \bar{\otimes} L(\mathbb{G})$.

$\Uparrow$ $\iff \exists$ unitary in $L(\mathbb{X})$.

$\mathbb{G}$ C*-cleft if $\mathcal{G}^u = \mathcal{G}^u_{\hat{\Omega}^u}$, $\hat{\Omega}^u \in M(C^*(\mathbb{G}) \otimes C^*(\mathbb{G}))$

$\iff \exists$ unitary in $M(C^*(\mathbb{X}))$.

Examples:

1. Not all $\mathcal{G}$ W*-cleft, e.g. $\mathbb{G} = \widehat{SU_q(2)}$ (Bichon - De Rijdt - Vaes '06)
2. $\mathbb{G}$ compact $\Rightarrow$ all $\mathcal{G}$ W*-cleft (DC - Martos - Mest '24)
3. $\mathcal{G}$ W*-cleft $\not\Rightarrow$ $\mathcal{G}$ C*-cleft

   (e.g. classical central extensions, $\underset{\substack{\cong \\ \mathbb{T}}}{\{\pm 1\}} \hookrightarrow SU(2) \twoheadrightarrow SO(3)$)

$\mathbb{G}$ linking q.grpd. $\Rightarrow$ unitary antipode $\hat{R}^u_{\mathbb{G}} : \begin{pmatrix} C^*(\mathbb{H}) & C^*(\mathbb{X}) \\ C^*(\mathbb{Y}) & C^*(\mathbb{G}) \end{pmatrix} \to \begin{pmatrix} C^*(\mathbb{H}) & C^*(\mathbb{Y}) \\ C^*(\mathbb{X}) & C^*(\mathbb{G}) \end{pmatrix}$

$\mathbb{G}$ C^*-cleft $\Rightarrow \hat{R}^u_{\mathbb{G}} \begin{pmatrix} w & x \\ y & z \end{pmatrix} = \begin{pmatrix} \hat{X}_u \hat{R}^u(w) \hat{X}_u^* & \hat{X}_u \hat{R}^u(y) \\ \hat{R}^u(x) \hat{X}_u^* & \hat{R}^u(z) \end{pmatrix}$, $x, y, w, z \in C^*(\mathbb{G})$

$\hat{R}^u = \hat{R}^u_{\mathbb{G}}$

Thm: $\mathbb{G}$ LCQG, $\mathbb{G}$ C^*-cleft quantum linking groupoid.

① $(id, u) : \mathrm{Rep}_u(\mathbb{G}) \underset{\otimes}{\cong} \mathrm{Rep}_u(\mathbb{H})$

② IF $\mathbb{G} \overset{\alpha}{\curvearrowright} M$:

$$\Rightarrow \begin{cases} L^2(\mathrm{Ind}_{\mathbb{X}}(M)) \cong L^2(M) \\ J_{\mathrm{Ind}_{\mathbb{X}}(M)} \cong \pi_{st}(\hat{X}_u) J_M \\ "U_{\mathrm{Ind}_{\mathbb{X}}(\alpha)} \cong U_\alpha" \end{cases}$$

Left standard rep. $C^*(\mathbb{H})$
$\cong$ $\quad$ $=$
Left standard rep. $C^*(\mathbb{G})$

Question: $\mathbb{G}$ C^*-cleft and $L(\mathbb{G}) \cong L(\mathbb{G}_{\hat{\Omega}})$

$\Longrightarrow$ $\hat{\Omega}$ lifts to $\hat{\Omega}_u \in M(C_u^*(\mathbb{G}) \otimes C_u^*(\mathbb{G}))$

$\leadsto$ $\hat{\Omega}_u \to \hat{\Omega}$ under reduction?

No: e.g. $\hat{\Omega} = (u^* \otimes u^*)\hat{\Delta}(u)$, $\quad u \in L(\mathbb{G})$

If **YES**: $\hat{\Omega}$ admits **universal lift** $\hat{\Omega}_u$.

Also: if exists, then not unique (unless $C^*(\mathbb{G}) = C^*_{red}(\mathbb{G})$)

Problem: $\mathbb{G} = \mathbb{G}_{\Omega}$ W^*-cleft and $\exists$ unitary 2-cocycle lift $\hat{\Omega}_u \to \hat{\Omega}$

$\overset{?}{\Longrightarrow}$ $\mathbb{G}$ C^*-cleft?

$\leadsto$ We do not know how to link

$$\left(C^*(\mathbb{G}), \hat{\Delta}_u\right) \longleftrightarrow \left(\begin{pmatrix} C^*(\mathbb{G}) & C^*(\mathbb{G}) \\ C^*(\mathbb{G}) & C^*(\mathbb{G}) \end{pmatrix}, \begin{pmatrix} \hat{\Omega}_u \hat{\Delta}_u(-)\hat{\Omega}_u^* & \hat{\Omega}_u \hat{\Delta}_u(-) \\ \hat{\Delta}_u(-)\hat{\Omega}_u^* & \hat{\Delta}_u \end{pmatrix}\right)$$

Thm: If $\hat{\mathcal{X}} \in L(\mathbb{G}_1) \bar{\otimes} L(\mathbb{G}_2)$ is unitary bicharacter, $\mathbb{G} = \mathbb{G}_1 \times \mathbb{G}_2$

then $\hat{\Omega} = \hat{\mathcal{X}}_{32} \in L(\mathbb{G}) \bar{\otimes} L(\mathbb{G})$ admits universal lift

$$\hat{\Omega}_u = \hat{\mathcal{X}}^u_{32} \in C^*(\mathbb{G}) \otimes C^*(\mathbb{G}).$$

Remark: $C^*(\mathbb{G}) = C^*(\mathbb{G}_1) \otimes_{max} C^*(\mathbb{G}_2)$

Pf: Strategy:

1. Show $\mathbb{G}_{\hat{\Omega}}$ is C^*-deft:

$$\begin{tikzcd} & M_2(C^*(\mathbb{G}_1) \otimes_{max} C^*(\mathbb{G}_2)) \arrow[d, equal] \\ C^*(\mathbb{G}_{\hat{\Omega}}) \arrow[ur, "(\hat{\pi}_1 \otimes_{max} \hat{\pi}_2)\hat{\Delta}_u^{max}"] \arrow[r, "\cong", "\text{as } C^*\text{-alg.}"'] & C^*(\mathbb{G}^{triv}) \end{tikzcd}$$

2. Show that resulting unitary 2-cocycle on $C^*(\mathbb{G})$ has the form $\tilde{\Omega}_u = \tilde{\mathcal{X}}^u_{32}$

3. Show that $\tilde{\mathcal{X}}_u = (Ad(u) \otimes Ad(u))(\hat{\mathcal{X}}_u)$, grouplike $u \in M(C^*(\mathbb{G}))$, and deduce $\hat{\Omega}$ admits lift $\hat{\Omega}_u$. ∎

Application: $\mathbb{G}$ LCQG $\Rightarrow$ $C_0^u(D\mathbb{G}) = C_0^u(\mathbb{G}) \otimes_{max} C_0^u(\hat{\mathbb{G}})$.